9/2020

B.Sc. Part 2

# Infinite series (contd.) (3rd Paper)

## Raabe's test (contd.)

**Q.** Test the convergence of the series

$$x^2(\log 2)^q + x^3(\log 3)^q + x^4(\log 4)^q + \cdots \text{ to } \infty.$$

**Soln.** Here, the $n^{th}$ term of the series $= x^{n+1}[\log(n+1)]^q$

i.e. $u_n = x^{n+1}[\log(n+1)]^q$

$$\Rightarrow u_{n+1} = x^{n+2}[\log(n+2)]^q$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{x^{n+1}}{x^{n+2}} \frac{[\log(n+1)]^q}{[\log(n+2)]^q} = \frac{1}{x}\left[\frac{\log(n+1)}{\log(n+2)}\right]^q$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{1}{x}\left[\frac{\log(n+2) + \{\log(n+1) - \log(n+2)\}}{\log(n+2)}\right]^q$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{1}{x}\left[\frac{\log(n+2) + \log\left(\frac{n+1}{n+2}\right)}{\log(n+2)}\right]^q$$

$$= \frac{1}{x}\left[1 + \frac{1}{\log(n+2)} \cdot \log\left(\frac{n+1}{n+2}\right)\right]^q$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{1}{x}\left[1 + \frac{1}{\log(n+2)} \cdot \log\left(\frac{(n+2)-1}{n+2}\right)\right]^q$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{1}{x}\left[1 - \frac{1}{\log(n+1)} \log\left(1 - \frac{1}{n+1}\right)\right]^p$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \infty\right)$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{1}{x}\left[1 + \frac{1}{\log(n+1)}\left\{\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \cdots \infty\right\}\right]^p$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{1}{x}\left[1 + \frac{1}{\log(n+1)}\left\{\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \cdots \infty\right\}\right]^p$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{1}{x}\left[1 + \frac{1}{(n+1)\log(n+1)} + \frac{1}{2(n+1)^2\log(n+1)} + \frac{1}{3(n+1)^3\log(n+1)} + \cdots \infty\right]^p$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{1}{x}\left[1 + \frac{p}{(n+1)\log(n+1)} + \frac{p}{2(n+1)^2\log(n+1)} + \frac{p}{3(n+1)^3\log(n+1)} + \cdots \infty\right]$$

Take $\underset{n\to\infty}{Lt}$ in both sides of (1), we've

$$\underset{n\to\infty}{Lt}\frac{U_n}{U_{n+1}} = \frac{1}{x}\cdot\underset{n\to\infty}{Lt}\left[1-\frac{q}{(n+2)\log(n+2)}-\frac{q}{2(n+2)^2\log(n+2)}-\frac{q}{3(n+2)^3\log(n+2)}\cdots\text{to}\infty\right]$$

$$\Rightarrow \underset{n\to\infty}{Lt}\frac{U_n}{U_{n+1}} = \frac{1}{x}\left[1-0-0-0-0\right]=\frac{1}{x}$$

So, the given series $\sum U_n$ is convergent if $\frac{1}{x}>1$, i.e. $x<1$

and $\sum U_n$ is divergent if $\frac{1}{x}<1$ i.e. $x>1$.

But this test fails when $\frac{1}{x}=1$ i.e. $x=1$.

Then, from (1), when $x=1$

$$\frac{U_n}{U_{n+1}} = 1-\frac{q}{(n+2)\log(n+2)}-\frac{q}{2(n+2)^2\log(n+2)}-\frac{q}{3(n+2)^3\log(n+2)}-\cdots\text{to}\infty$$

$$\Rightarrow \frac{U_n}{U_{n+1}} - 1 = -\frac{q}{(n+2)\log(n+2)} - \frac{q}{2(n+2)^2\log(n+2)} - \frac{q}{3(n+2)^3\log(n+2)} - \cdots \text{ to } \infty$$

$$\Rightarrow n\left(\frac{U_n}{U_{n+1}} - 1\right) = -q\left[\frac{n}{(n+2)\log(n+2)} + \frac{n}{2(n+2)^2\log(n+2)} + \frac{n}{3(n+2)^3\log(n+2)} + \cdots \text{ to } \infty\right]$$

$$\Rightarrow n\left(\frac{U_n}{U_{n+1}} - 1\right) = -q\left[\frac{1}{(1+\frac{2}{n})\log(n+2)} + \frac{1}{2(1+\frac{2}{n})(n+2)\log(n+2)} + \cdots \text{ to } \infty\right]$$

Take $\underset{n\to\infty}{Lt}$ both sides, we get

$$\underset{n\to\infty}{Lt}\; n\left(\frac{U_n}{U_{n+1}} - 1\right) = -q\left[0+0+0\right] = 0$$

$$\Rightarrow \underset{n\to\infty}{Lt}\; n\left(\frac{U_n}{U_{n+1}} - 1\right) = 0 < 1$$

So, the given series is divergent when $x=1$.

Hence, the series is convergent when $x < 1$ and divergent when $x \geq 1$.